\( \pi \) rears its head, again and again

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1 \( \pi \) rears its head in geometry (unsurprisingly)

In Euclidean geometry the circumference of a circle is \( \pi \) times its diameter.

\[
\pi = \frac{C}{d}
\]

Equivalently

\[
\pi = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}
\]

This integral is just the arc length of a unit semicircle. And of course the area of a circle is \( \pi \) times the square of the radius.

\[
\pi = \frac{A}{r^2}
\]

2 \( \pi \) rears its head in trigonometry

The most obvious trig identity you might use to compute digits of \( \pi \) is

\[
\pi = 4 \arctan 1
\]

Using the Taylor series

\[
\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots
\]

this gives us:

\[
\pi = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \right)
\]

but this is really slow to converge because each term cancels out most of the previous term. It takes about half a million terms to get 5 digits of precision. But there are other trig identities that give us faster convergence. This one is due to John Machin (1706):

\[
\pi = 4 \left( 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \right)
\]
Its series expansion converges at a more useful rate, about 1.4 digits per term. Before the advent of computers Machin’s formula was used by many (human) calculators to calculate more and more digits of $\pi$. Machin himself computed the first 100. Daniel Ferguson (1946) got 620 digits, the best result achieved without a digital computer.

3 \hspace{1em} \pi \text{ rears its head in an infinite product}

In 1655, John Wallis discovered this infinite product:

\[ \frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{4}{7} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \ldots \]

4 \hspace{1em} \pi \text{ rears its head in continued fractions}

\[ \pi = \frac{4}{1 + \frac{1^2}{1 + \frac{2^2}{1 + \frac{3^2}{1 + \frac{4^2}{1 + \ldots}}}}} \]

and

\[ \frac{\pi}{2} = 1 - \frac{1}{3 - \frac{2 \cdot 3}{1 - \frac{1 \cdot 2}{4 \cdot 5}}} \]

The ‘simple’ (numerators all 1) continued fraction for $\pi$ is

\[ \pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{292 + \frac{1}{1 + \ldots}}}} \]

Billions of terms of the simple continued fraction have been computed without any meaningful pattern being discovered.
5 \( \pi \) rears its head in complex analysis

Let’s look at Euler’s famous equation:

\[ e^{i\pi} + 1 = 0 \]

The easiest way to see that this is true is to look at the Maclaurin series for \( e^{i\theta} \) and separate the even and odd terms:

\[
e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n=0}^{\infty} i^{2n} \frac{\theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i^{2n+1} \frac{\theta^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} = \cos \theta + i \sin \theta
\]

(You have to be careful about convergence when you do infinite rearrangements of infinite sums, but everything is OK in this case.) Substituting \( \theta = \pi \) we get \( e^{i\pi} = -1 + 0i \). Bingo.

6 \( \pi \) rears its head in number theory

In 1644, Pietro Mengoli proposed the problem of evaluating

\[ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \]

Euler got the answer in 1734: \( \frac{\pi^2}{6} \). This is important because it’s a value \( \zeta(2) \) of the Riemann zeta function:

\[
\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \quad \text{Re}(s) > 1
\]

You can prove that the sum and the product are the same by first seeing that

\[
\frac{1}{1 - p^{-s}} = \sum_{k=0}^{\infty} \frac{1}{p^{ks}}
\]

(by long division) and then

\[
\prod_{p \text{ prime}} \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \sum_{n=0}^{\infty} \frac{1}{n^s}
\]

Running the distributive law on the LHS (an infinite number of times!) produces a sum containing each term of the RHS exactly once because of unique prime factorization.
Because the infinite product for \( \zeta(s) \) has a factor for each prime number, it is of central importance in analytic number theory.

Evaluating the Riemann zeta function at any even positive integer yields a power of \( \pi \) times a rational number. For example:

\[
\begin{align*}
\zeta(4) & = \frac{\pi^4}{90} \\
\zeta(6) & = \frac{\pi^6}{945} \\
\zeta(8) & = \frac{\pi^8}{9450}
\end{align*}
\]

Surprisingly, almost nothing is known about values of \( \zeta \) at odd positive integers.

7 \( \pi \) rears its head in statistics

This integral, due to Gauss:

\[
\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}
\]

is important for all sorts of reasons. In particular the Standard Normal probability distribution, whose probability density function is

\[
N(0, 1) = \frac{e^{-x^2}}{\sqrt{\pi}}
\]

is central to probability and statistics.

Look at this integral:

\[
\pi = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}
\]

This means that

\[
\frac{1}{\pi(x^2 + 1)}
\]

is a probability density function (for the Cauchy distribution). The Cauchy distribution has the interesting feature that it has no mean, because the integral

\[
\int_{-\infty}^{\infty} \frac{x}{\pi(x^2 + 1)} dx
\]

diverges.